Homological algebra exercise sheet Week 11

1. (**Mapping cylinder**) Let $\mathcal{A}$ be an abelian category and consider a morphism $f : B \to C$ between cochain complexes in B. We define the mapping cylinder of f to be the cochain complex $\mathrm{cyl}(f)$ given by

   - $\mathrm{cyl}(f)^n := B^n \oplus B^{n+1} \oplus C^n$,
   - differential maps $d = \mathrm{cyl}(f)^n \to \mathrm{cyl}(f)^{n+1}$ given by the matrix

   $$\begin{bmatrix} d_B & \mathrm{id}_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{bmatrix}.$$

   (a) Show that $\mathrm{cyl}(f)$ is a well-defined cochain complex.

   In the following, we denote by $\mathrm{cyl}(B)$ the mapping cylinder of the identity map id_B on the object B.

   (b) Given $f, g : B \to C$, show that f and g are chain homotopic if and only if there exists a family of maps $\{s^n : B^{n+1} \to C^n\}$ such that $(f, s, g) : \mathrm{cyl}(B) \to C$ is a morphism of cochain complexes.

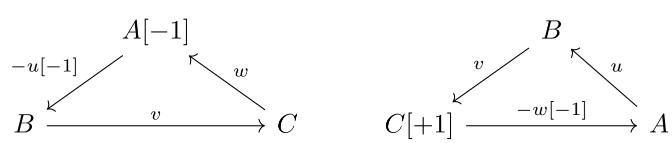
   (c) Consider the maps $\alpha : B \to \mathrm{cyl}(B)$, given by $\alpha(b) = (b, 0, 0)$, and $\beta : \mathrm{cyl}(B) \to B$, given by $\beta(b', b'', b) = b' + b$. Show that the map $s(b', b'', b) = (0, b, 0)$ defines a chain homotopy between $\mathrm{id}_{\mathrm{cyl}(B)}$ and $\alpha\beta$. Deduce that α is a chain homotopy equivalence.

   (d) Now consider $\alpha' : B \to \mathrm{cyl}(B)$, given by $\alpha'(b) = (0, 0, b)$. Using the universal property of the category $\mathbf{K}(\mathcal{A})$, deduce that α' is also a chain homotopy equivalence. Find an explicit map s defining a chain homotopy between $\mathrm{id}_{\mathrm{cyl}(B)}$ and $\alpha'\beta$. (Hint: use $F : \mathbf{Ch}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$.)

2. (**Examples and exact triangles**) Consider an abelian category $\mathcal{A}$ and objects A, B and C in $\mathcal{A}$.

   (a) Find exact triangles for the maps 0_A and 1_A. (Hint: use the mapping cones in the category $\mathbf{K}(\mathcal{A})$.)

   (b) Given an exact triangle (u, v, w) on A, B, C, show that the rotates



are also exact triangles.

3. Show that there is no morphism $w : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ making the short exact sequence
$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 1} \mathbb{Z}/2\mathbb{Z} \to 0$$
into an exact triangle $(2, 1, w)$.

4. Let $\mathcal{D}$ be a triangulated category and let the two rows below be two exact triangles, and $g : B \to B'$ a morphism. Show that if $v'gu = 0$ then there exists $f : A \to A', h : C \to C'$ such that (f, g, h) is a morphism of triangles (i.e. the diagram below commutes).
$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & TA' \end{array}$$
Hint: consider cohomological functor $\mathrm{Hom}(X, \cdot)$.

5. Let $\mathcal{A}^{\mathbb{Z}}$ be the category of graded objects in an abelian category $\mathcal{A}$ (i.e. $\{A_n\}_{n\in\mathbb{Z}}$ with A_n an object in $\mathcal{A}$), a morphism form $A = \{A_n\}$ to $B = \{B_n\}$ being a family of morphisms $f_n : A_n \to B_n$. Define $T(A)$ to be the translated graded object such that $T(A)_n = A_{n-1}$ and call (u, v, w) an exact triangle on (A, B, C) if for all n the sequence
$$A_n \xrightarrow{u} B_n \xrightarrow{v} C_n \xrightarrow{w} A_{n-1} \xrightarrow{u} B_{n-1}$$
is exact in $\mathcal{A}$.
   (a) If $\mathcal{A} = \mathbf{Ab}$ the category of abelian group, show that axiom (TR1) and (TR2) hold but (TR3) fails for $\mathcal{A}^{\mathbb{Z}}$. Hint: consider group extension.
   (b) If $\mathcal{A}$ is the category of vector space over a field, show that $\mathcal{A}^{\mathbb{Z}}$ is a triangulated category.

6. Show that in a triangulated category, every commutative square in the upper diagram below can be completed to the lower diagram, in which all the rows and columns are exact triangles and all the squares commute, except the bottom right corner anticommutes.
$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle u}\downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

$$
\begin{array}{ccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & T(A) \\
{\scriptstyle u}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Tu} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & T(A') \\
{\scriptstyle v}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Tv} \\
A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & T(A'') \\
{\scriptstyle w}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Tw} \\
T(A) & \xrightarrow{Ti} & T(B) & \xrightarrow{Tj} & T(C) & \xrightarrow{Tk} & T^2(A)
\end{array}
$$

Hint: use (TR1) to construct everything except the third column, and construct an exact triangle on (A, B', D), then use (TR4) to construct exact triangles on (C, D, B''), (A'', D, C'), and finally (C', C'', C).